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Adjoint Method for Missile Performance Analysis on State-Space Models

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The adjoint method is a method widely used in the preliminary design of guidance loops to obtain quick estimates of the performance of a guided weapon while avoiding time consuming Monte Carlo simulation experiments. Traditionally, the adjoint method is presented as a set of rules for transforming the original (linearized) model of the guidance loop to an adjoint model that is used to obtain the performance estimate for the original model in a single simulation run. An attempt is made to derive the adjoint method in the general setting of state-space models. This does not only lead to an elegant and clear exposition of the adjoint method, but it also extends the area of application of the adjoint method to more general situations that were not previously covered in the literature. One possible extension is illustrated with the analysis of the performance of a rolling missile against a target that performs a maneuver with a random start time.

I. Introduction

THE adjoint method has a well-established place in the arsenal of tools available for guidance-loop performance analysis, in particular in preliminary phases of guided missile design. The success of the method is mainly because it is relatively simple to use and because it enables a quick performance assessment for a wide range of engagement conditions. The method as a general analysis method for control systems under stochastic disturbances was introduced by Laning and Battin.[1] However, some authors, such as Zarchan,[2] place the origin of the method back to the work of the 19th century mathematician Vitto Voltera. Zarchan mentions the application of the method in computing ballistic dispersions as early as the 1920s. Although the method is also applicable for the analysis of systems under deterministic disturbances, it is the stochastic case that demonstrates the real power of the method. In this case, the method can be efficiently used to avoid time consuming Monte Carlo simulations when linear approximation results can deliver sufficient accuracy.

The traditional presentation of the adjoint method in the literature[2,3] is based on the input–output (transfer function-type) system representation with a strong emphasis on the procedural aspects of the method: inverting the sense of the signal flow, substituting time with time-to-go, etc.

In this work, we pursue in detail the derivation of the construction rules for the adjoint system based on state-space models. We consider both the deterministic and the stochastic case for continuous-time models. In the stochastic case, our approach is shown to extend the potential of the method beyond applications currently reported in the literature that assume uncorrelated inputs and look only at the variance of the chosen output signal. We show here that the adjoint method can perfectly accommodate correlations between the different inputs and that a possible outcome is the influence of the random inputs on the covariance of two different outputs at the a priori fixed moment of time. In this sense, it is shown that the analysis power of the adjoint method is almost identical to that of another method for preliminary design analysis called the covariance matrix method. For a presentation of the covariance matrix method, the interested reader may consult Ref. 2. A comparison between these two popular methods for weapon performance analysis is given in Ref. 4.

The idea to formulate the adjoint method for state-space equations is not new. In fact, the proof of the method in Ref. 1 is done in terms of state-space equations, although not in the most general case, and the principles of the method are formulated in block diagram terms. In Ref. 5, Sec. 2.VI, we also find a special formulation of the adjoint method in terms of state-space equations. However, as far as we know, there is no systematic presentation of the method in state-space context.

The essence of the adjoint method is to deduce the separate effects of each input on the value of one output at a fixed moment of time by solving an initial-value problem for the so-called adjoint model. This means that the adjoint method is fundamentally limited to linear, possibly time-varying, models because they satisfy the superposition principle. Otherwise, it is impossible to speak about separate effects of the different inputs. In particular, if the method is to be applied to a nonlinear phenomenon such as the guidance loop of a missile, it is necessary to linearize first the model around a nominal, desired flight profile. The resulting linear, time-varying system is used to apply the adjoint method.

An important feature in our presentation is the distinction between the dual system and the adjoint response. The dual system is a system associated with the original system at a given moment of time. The adjoint response is defined as a particular response of the dual system. The notion of dual system and its properties related to system interconnections accounts for the success of the adjoint method as a practical method applied on block diagram representations of system models. The adjoint response is the specific output of the adjoint method that allows for a quick assessment of the system performance. This is a slight departure from the standard terminology in the guidance analysis literature, but it conforms to usual terminology in control systems literature and it adds, in our opinion, to the clarity of the exposition.

This paper is structured as follows. In Sec. II, the mathematical framework for the adjoint method in the case of deterministic continuous time is presented. In Sec. III, we consider the stochastic case. Finally, in Sec. IV, we illustrate the theoretical developments with an application to the analysis of the miss distance due to a lateral target maneuver. First, we consider the case of a roll-stabilized missile. Second, we consider the case of a rolling missile, demonstrating the effect of cross coupling on the distribution of the miss distance vector. The discrete-time counterpart of the theory developed in this paper is presented in Appendix A.

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II. State-Space Formulation of the Adjoint Method in the Deterministic Case

Let us consider a linear time-varying system G with m inputs and p outputs:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \qquad \boldsymbol{y}(t) = C(t)\boldsymbol{x}(t) \tag{1}$$

with initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{2}$$

Here $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are assumed to be matrix-valued smooth functions with $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times m}$, and $C(t) \in R^{p\times n}$.

Let us also fix a moment of time $t_f > t_0$ as being the time at which the output of the system is particularly interesting. Following current terminology in the control systems literature, for example, as in Ref. 6, Sec. 1.8, we define the dual of system (1) at time t_f as the linear time-varying system:

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}^\diamond = A^T(t_f - t_g)\boldsymbol{x}^\diamond + C^T(t_f - t_g)\boldsymbol{u}^\diamond$$

$$\boldsymbol{y}^\diamond(t_g) = B^T(t_f - t_g)\boldsymbol{x}^\diamond(t_g) \tag{3}$$

We denote by $G^\diamond$ the dual of the system G. Notice that the independent time variable in the case of the dual system was denoted by t_g to stress that it is a different variable than the time variable t of the original system.

Assume that system (1) has a single output, $p = 1$. We define the adjoint response of the system (1) at time t_f as the free response ($\boldsymbol{u}^\diamond \equiv 0$) of the dual system (3) with initial condition

$$\boldsymbol{x}^\diamond(0) = C^T(t_f) \tag{4}$$

We denote the state adjoint response $\boldsymbol{x}^{\mathrm{adj}}$ and the output adjoint response $\boldsymbol{y}^{\mathrm{adj}}$. By definition, they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}^{\mathrm{adj}} = A^T(t_f - t_g)\boldsymbol{x}^{\mathrm{adj}}, \qquad \boldsymbol{y}^{\mathrm{adj}}(t_g) = B^T(t_f - t_g)\boldsymbol{x}^{\mathrm{adj}}(t_g) \tag{5}$$

$$\boldsymbol{x}^{\mathrm{adj}}(0) = C^T(t_f) \tag{6}$$

In the general case, where system (1) has $p > 1$ output variables, it is possible to use the earlier definition for each output variable separately and define an adjoint response at time t_f for each one of the output variables.

The essence of the adjoint method is that the adjoint response can be used to relate the values of the initial variable and of the input of system (1) to its output at time t_f. In this section we will show how this principle works in the deterministic case. In the next section we will extend it to the stochastic case.

Before the introduction of the main result of this section, notice that the well-known rules for "constructing the adjoint" (Ref. 2, Chap. 3) are the consequence of the following elementary properties of the dual of a system:

1) Each system is the dual of its dual system.

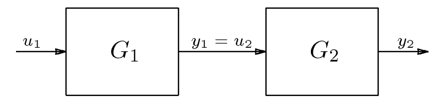
2) The dual of the series connection of G_1 with G_2 is the series connection of $G_2^\diamond$ with $G_1^\diamond$.

3) The dual of the parallel connection of G_1 and G_2 is the sum connection of $G_1^\diamond$ and $G_2^\diamond$.

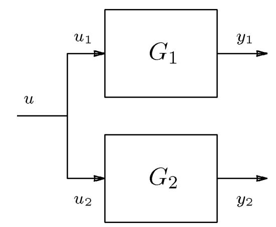
4) The dual of the sum connection of G_1 and G_2 is the parallel connection of $G_1^\diamond$ and $G_2^\diamond$.

(See Fig. 1 for the definition of the three basic system connections: the series connection, the parallel connection, and the sum connection.) The proof of these properties is a straightforward application of the definition of the dual system.

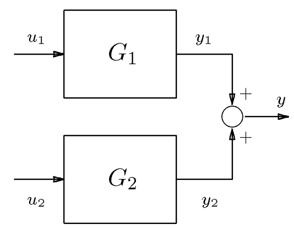
The following result is the basic ingredient of the application of the adjoint method to guidance-loop performance analysis in the noise-free case.



a) Series connection



b) Parallel connection



c) Sum connection

Fig. 1 Basic system connections of two systems G_1 and G_2.

Proposition 1. The final value of the output of system (1) with initial condition (2) is

$$y(t_f) = \left[\boldsymbol{x}^{\mathrm{adj}}(t_f - t_0)\right]^T \boldsymbol{x}_0 + \int_{t_0}^{t_f} \left[\boldsymbol{y}^{\mathrm{adj}}(t_f - \tau)\right]^T \boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{7}$$

where $\boldsymbol{x}^{\mathrm{adj}}$ and $\boldsymbol{y}^{\mathrm{adj}}$ are the state and the output adjoint response of system (1) at t_f, respectively.

Proof. Let us introduce the transition matrix of system (1), denoted by $\Phi(t, \tau)$ for $t, \tau \in [t_0, t_f]$. The transition matrix by definition satisfies the following conditions:

$$\frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau) \tag{8}$$

$$\Phi(t,\tau)\Phi(\tau,\sigma) = \Phi(t,\sigma) \tag{9}$$

$$\Phi(t,t) = I_n \tag{10}$$

where $t, \tau, \sigma \in [t_0, t_f]$ and I_n is the identity matrix of order n. It can be also deduced that the transition matrix satisfies

$$\frac{\partial}{\partial \tau}\Phi(t,\tau) = -\Phi(t,\tau)A(\tau) \tag{11}$$

The output of system (1) can be written with the variations-of-constants formula as

$$y(t_f) = C(t_f)\Phi(t_f,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t_f} C(t_f)\Phi(t_f,\tau)B(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{12}$$

Comparing this formula with formula (7), after an obvious change of the variables under integration, we see that it is sufficient to prove that

$$\boldsymbol{y}^*(\tau) \triangleq B^T(t_f - \tau)\Phi^T(t_f, t_f - \tau)C^T(t_f) \tag{13}$$

$$\boldsymbol{x}^*(\tau) \triangleq \Phi^T(t_f, t_f - \tau)C^T(t_f) \tag{14}$$

with $\tau \in [0, t_f - t_0]$ the output and the state of the adjoint system (5), respectively, with initial condition (4). In fact, if we notice that

$$\boldsymbol{y}^*(\tau) = B^T(t_f - \tau)\boldsymbol{x}^*(\tau)$$

it only remains to prove the assertion for $\boldsymbol{x}^*$. First, notice that, by Eq. (10),

$$\boldsymbol{x}^*(0) = \Phi^T(t_f, t_f)C^T(t_f) = C^T(t_f)$$

and the initial condition is satisfied. Taking now the derivative with respect to τ in formula (14) and applying Eq. (11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{x}^* = A^T(t_f - \tau)\Phi^T(t_f, t_f - \tau)C^T(t_f) = A^T(t_f - \tau)\boldsymbol{x}^*$$

and the proof is completed. □

The significance of Proposition 1 is that to assess the influence on the final value of the output of the initial condition and the input of the time-varying system, it is enough to find one initial-value solution of the adjoint system. Also notice that the influence of the input value at a time t is given by the value of the adjoint output at time $t_f - t$, or equivalently, the time-to-go till the observation moment.

Also note that the state-space formulation of the adjoint response differs essentially from the input–output formulation in that it does not involve any impulsive inputs. In this formulation, the impulsive inputs are replaced by the initial condition (6). In fact, this is also the way in which the method is used in numerical applications because it is easier to specify and to solve an initial-value problem than to introduce an impulsive input.

III. Stochastic Case

Let us assume that the input $\boldsymbol{u}$ of system (1) is a white noise signal of zero mean and power spectral density $V(t)$, that $\boldsymbol{x}_0$ is a stochastic variable of mean $\bar{\boldsymbol{x}}_0$ and covariance matrix Q_0, and that $\boldsymbol{x}_0$ is not correlated with $\boldsymbol{u}$. In this case, a classical result, for example, as in Ref. 6, Theorem 1.52, expresses the stochastic characteristics of the state and output response of system (1) as follows.

The solution $\boldsymbol{x}(t)$ of system (1) with initial condition (2) is, for each $t \geq t_0$, a stochastic variable with mean

$$\bar{\boldsymbol{x}}(t) = \Phi(t, t_0)\bar{\boldsymbol{x}}_0 \tag{15}$$

and covariance matrix $Q(t)$ satisfying the matrix differential equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + B(t)V(t)B^T(t) \tag{16}$$

with initial condition

$$Q(t_0) = Q_0 \tag{17}$$

Furthermore, the autocovariance matrix function of $\boldsymbol{x}(\cdot)$ is

$$R_x(t_1, t_2) = E[\boldsymbol{x}(t_1) - \bar{\boldsymbol{x}}(t_1)][\boldsymbol{x}(t_2) - \bar{\boldsymbol{x}}(t_2)]^T$$
$$= \Phi(t_1, t_0)Q_0\Phi^T(t_2, t_0) + \int_{t_0}^{\min(t_1,t_2)} \Phi(t_1, \tau)B(\tau)V(\tau) \times B^T(\tau)\Phi^T(t_2, \tau)\,\mathrm{d}\tau \tag{18}$$

and can also be written as

$$R_x(t_1, t_2) = \begin{cases} Q(t_1)\Phi^T(t_2, t_1), & t_2 \geq t_1 \\ \Phi(t_1, t_2)Q(t_2), & t_1 \geq t_2 \end{cases} \tag{19}$$

The output $\boldsymbol{y}(t)$ is a stochastic variable with mean

$$\bar{\boldsymbol{y}}(t) = C(t)\Phi(t, t_0)\bar{\boldsymbol{x}}_0 \tag{20}$$

and variance matrix

$$R(t) = C(t)Q(t)C^T(t) \tag{21}$$

The autocovariance matrix function of $\boldsymbol{y}(\cdot)$ is

$$R_y(t_1, t_2) = \begin{cases} C(t_1)Q(t_1)\Phi^T(t_2, t_1)C^T(t_2), & t_2 \geq t_1 \\ C(t_1)\Phi(t_1, t_2)Q(t_2)C^T(t_2), & t_1 \geq t_2 \end{cases} \tag{22}$$

These facts represent the mathematical background for the variance matrix method, for example, as in Ref. 2, Chap. 5, that is commonly suggested as an alternative to the adjoint method for performance prediction of a system with noisy inputs.

Essentially the variance matrix method consists of integrating the matrix differential equation (16) with initial condition (17) and computing the variance matrix of the output using formula (21).

Of course, the results cited before can be used directly only if $\boldsymbol{u}$ is a white noise signal in the (weak) sense that its values at different moments of time are uncorrelated. If this is not the case, the standard approach is to use a shaping filter, which is a linear time-varying system S with white noise input $\boldsymbol{v}$, such that its output has the same autocovariance function as the original input $\boldsymbol{u}$. Subsequently, the output covariance can be obtained by applying relations (16–22) to the series connection of the original system with the shaping filter.

The shaping filter method can be easily justified using the following expression of the auto-covariance matrix function of $\boldsymbol{x}(\cdot)$ that holds also in the case that the input noise is not white:

$$R_x(t_1, t_2) = E[\boldsymbol{x}(t_1) - \bar{\boldsymbol{x}}(t_1)][\boldsymbol{x}(t_2) - \bar{\boldsymbol{x}}(t_2)]^T$$
$$= \Phi(t_1, t_0)Q_0\Phi^T(t_2, t_0) + \int_{t_0}^{t_1}\mathrm{d}\tau_1\int_{t_0}^{t_2}\mathrm{d}\tau_2\big[\Phi(t_1, \tau_1)B(\tau_1) \times R_u(\tau_1, \tau_2)B^T(\tau_2)\Phi^T(t_2, \tau_2)\big] \tag{23}$$

where $R_u(t_1, t_2)$ represents the autocovariance matrix function of $\boldsymbol{u}(\cdot)$. It is easy to see that in the particular case that $R_u(\tau_1, \tau_2) = V(\tau_1)\delta(\tau_1 - \tau_2)$, corresponding to the case that $\boldsymbol{u}$ is white noise in the weak sense, expression (23) reduces to Eq. (18). For a slightly different formulation of these results, the reader may consult Theorem 1.47 in Ref. 6.

As we shall see in the sequel, these results can be used to deduce the stochastic version of the adjoint method. In fact, this approach leads to a true extension of the scope of the adjoint method for the stochastic case. The basic idea is to notice that system (16–21) can be regarded as a linear time-varying system with the state-vector represented by the matrix-valued function $Q(\cdot)$, the input represented by the matrix-valued function $V(\cdot)$, and the output represented by the matrix-valued function $R(\cdot)$.

If we assume for the moment that there is only one output variable of interest, then R represents the variance of this output variable. In case we are only interested in the variance of the output at a given time t_f, we can simply apply Proposition 1 to system (16–21).

In general, the off-diagonal coefficients of R are the covariances of the respective output pairs. Any of these coefficients can be considered as the output for the state-space equations (16) and Proposition 1 can be applied to obtain the covariance of two output variables at a given time t_f. The following result makes the idea explicit.

Proposition 2. Let us consider the linear time-varying system with two output signals:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \qquad y_1(t) = C_1(t)\boldsymbol{x}(t)$$
$$y_2(t) = C_2(t)\boldsymbol{x}(t) \tag{24}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ take values in $R^{1\times n}$ and $\boldsymbol{u}$ is a white noise signal of zero mean and power spectral density $V(t)$. Assume also that the initial value of the system

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{25}$$

is a stochastic vector of zero mean and covariance matrix Q_0.

The covariance of $y_1(t_f)$ and $y_2(t_f)$ is

$$\sigma_{y_1y_2}(t_f) = \operatorname{tr} P(t_f - t_0)Q_0 + \int_{t_0}^{t_f}\operatorname{tr}\big[B^T(\tau)P(t_f - \tau)B(\tau)V(\tau)\big]\,\mathrm{d}\tau \tag{26}$$

where $P(t)$ is an $n \times n$ matrix function satisfying the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t_g} P = A^T(t_f - t_g)P + PA(t_f - t_g) \tag{27}$$

with the initial condition

$$P(0) = C_1^T(t_f)C_2(t_f) \tag{28}$$

In particular, if $y_1 = y_2 = y$, that is, $C_1(t) = C_2(t)$ for all t, formula (26) gives the expression of the variance of the output variable y.

In the case that the power spectral density $V(t)$ is constant in time, formula (26) becomes

$$\sigma_{y_1 y_2}(t_f) = \mathrm{tr}\, P(t_f - t_0)\boldsymbol{Q}_0 + \mathrm{tr}\left[\boldsymbol{V}\int_{t_0}^{t_f} B^T(\tau)P(t_f - \tau)B(\tau)\,\mathrm{d}\tau\right] \tag{29}$$

Proof. Differential equation (16) satisfied by the covariance matrix of the state vector is a linear differential equation with state vector $\boldsymbol{Q}(t)$ and input vector $\boldsymbol{V}(t)$. If the two output variables y_1 and y_2 are regarded as the components of a two-dimensional output vector $\boldsymbol{y}(t) = [y_1(t) \;\; y_2(t)]^T$, then the covariance matrix of $\boldsymbol{y}(t)$ is, according to Eq. (21),

$$R(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} \boldsymbol{Q}(t)\begin{bmatrix} C_1^T(t) & C_2^T(t) \end{bmatrix}$$

The off-diagonal element of this symmetric matrix is the covariance of y_1 and y_2:

$$\sigma_{y_1 y_2}(t) = C_1(t)Q(t)C_2^T(t) \tag{30}$$

The last relation can be regarded as a linear output equation for the state equation (16). It is possible to write these equations in conventional form (1) by taking the state vector $\boldsymbol{x}$ to be the column stacked form of the matrix Q, the input vector $\boldsymbol{u}$ to be the column stacked form of the matrix V, and the output to be $\sigma_{y_1 y_2}$, and by defining appropriately the system matrices A, B, and C. In fact, these system matrices can be written explicitly using the Kronecker matrix product that is defined, in block matrix form, as

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \dots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

where $\{a_{ij}\}_{i=\overline{1,m}, j=\overline{1,n}}$ are the coefficients of A. If we denote the column-stacked form of a matrix X by stack$\{X\}$, that is,

$$\mathrm{stack}\{X\} = [x_{11} \;\; \dots \;\; x_{m1} \;\; x_{12} \;\; \dots \;\; x_{m2} \;\; \dots \;\; x_{1n} \;\; \dots \;\; x_{mn}]^T$$

then it can be easily shown that the Kronecker product has the following property:

$$\mathrm{stack}\{AXB\} = [B^T \otimes A]\,\mathrm{stack}\{X\}$$

When this property is used repeatedly, and when, for simplicity, stack$\{Q\}$ is denoted by $\boldsymbol{q}$ and stack$\{V\}$ is denoted by $\boldsymbol{v}$, system (16), (17), (30) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \left[A(t) \otimes I_n + I_n \otimes A^T(t)\right]\boldsymbol{q} + B(t) \otimes B(t)\boldsymbol{v} \tag{31}$$

$$\sigma_{y_1 y_2} = C_1(t) \otimes C_2(t)\boldsymbol{q} \tag{32}$$

The adjoint response of system (31) and (32) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{p} = \left[A^T(t_f - t_g) \otimes I_n + I_n \otimes A(t_f - t_g)\right]\boldsymbol{p}$$

$$\boldsymbol{p}(0) = C_2^T(t_f) \otimes C_1^T(t_f), \qquad \boldsymbol{v}^\diamond = B^T(t_f - t_g) \otimes B^T(t_f - t_g)\boldsymbol{p}$$

It is easy to see that unstacking the n^2 vector $\boldsymbol{p}$, that is, taking the matrix P such that stack$\{P\} = \boldsymbol{p}$, the matrix P satisfies Eq. (27) and condition (28). According to Proposition 1, we can write the final value of $\sigma_{y_1 y_2}$ as

$$\sigma_{y_1 y_2}(t_f) = \boldsymbol{p}^T(t_f - t_0)\boldsymbol{q}(0) + \int_{t_0}^{t_f} [\boldsymbol{v}^\diamond(t_f - \tau)]^T \boldsymbol{v}(\tau)]\,\mathrm{d}\tau$$

Now with the use of

$$\mathrm{tr}\, A^T B = [\mathrm{stack}\{A\}]^T \mathrm{stack}\{B\}$$

we deduce relation (26). □

Let us show that Proposition 2 contains the well-known formulation of the stochastic adjoint method. For this purpose, we reformulate the earlier result in terms of the adjoint response at time t_f of the original system (1).

Proposition 3. Consider system (1) with a single output variable [$C(t)$ is in $R^{1\times n}$]. The variance of the output at time t_f is

$$\sigma_y^2(t_f) = \left[\boldsymbol{x}^{\mathrm{adj}}(t_f - t_0)\right]^T Q_0 \boldsymbol{x}^{\mathrm{adj}}(t_f - t_0) + \int_{t_0}^{t_f} \left[\boldsymbol{y}^{\mathrm{adj}}(t_f - \tau)\right]^T V(\tau)\boldsymbol{y}^{\mathrm{adj}}(t_f - \tau)\,\mathrm{d}\tau \tag{33}$$

where $\boldsymbol{x}^{\mathrm{adj}}$ and $\boldsymbol{y}^{\mathrm{adj}}$ are the state and the output adjoint response of system (1) at t_f.

Proof. We notice that the solution of Eq. (27) with initial value $P(0) = C^T(t_f)C(t_f)$ is

$$P(t_g) = \Phi^T(t_f, t_f - t_g)C^T(t_f)C(t_f)\Phi(t_f, t_f - t_g) \tag{34}$$

where Φ is the transition matrix function of the system (1) that was introduced in the proof of Proposition 1. Indeed, the initial condition is clearly satisfied, and taking the derivative after t_g in solution (34), and using conditions (8) and (11), we immediately deduce that Eq. (27) is satisfied.

Substituting solution (34) in Eq. (26) for the case $y_1 \equiv y_2 \equiv y$ and using the fact that $\mathrm{tr}\, AB = \mathrm{tr}\, BA$, we deduce that

$$\sigma_y^2(t_f) = C(t_f)\Phi(t_f, t_0)Q_0\Phi^T(t_f, t_0)C^T(t_f) + \int_{t_0}^{t_f} \left[C(t_f)\Phi(t_f, \tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_f, \tau)C^T(t_f)\right]\mathrm{d}\tau$$

Now formula (33) follows from the formulas (14) and (13), which represent, as shown in the proof of Proposition 1, the state and the output adjoint responses, respectively. □

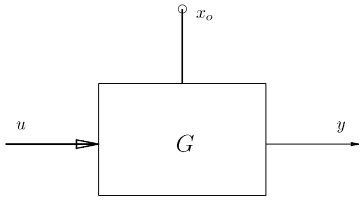
Proposition 3 is a generalization of the well-known rule for the construction of the stochastic adjoint model. The application of the (generalized) adjoint method for stochastic systems is shown in Fig. 2. In the case where the input signals are uncorrelated and they are normalized such that the matrix function $V(\cdot)$ is constant and equal to the identity matrix, then the quadratic form (Quad Form in Fig. 2) is simply the sum of squares of the adjoint response output signals and we recover the standard procedure as presented, for example, in Ref. 2.

When again Proposition 2 is used, it is possible to derive an expression of the covariance of two different output signals at a fixed time moment in terms of the adjoint responses.

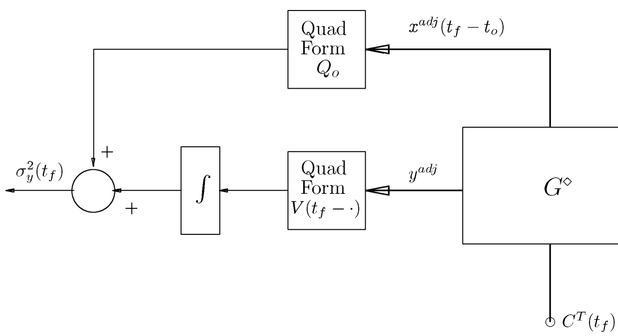
Proposition 4. The covariance of the two output signals of the system (24) at time t_f is

$$\sigma_{y_1, y_2}(t_f) = \left[\boldsymbol{x}_1^{\mathrm{adj}}(t_f - t_0)\right]^T Q_0 \boldsymbol{x}_2^{\mathrm{adj}}(t_f - t_0) + \int_{t_0}^{t_f} \left[\boldsymbol{y}_1^{\mathrm{adj}}(t_f - \tau)\right]^T V(\tau)\boldsymbol{y}_2^{\mathrm{adj}}(t_f - \tau)\,\mathrm{d}\tau \tag{35}$$

where $\boldsymbol{x}_1^{\mathrm{adj}}$ and $\boldsymbol{y}_1^{\mathrm{adj}}$ are the state and the output adjoint response, respectively, for the initialization $\boldsymbol{x}_1^{\mathrm{adj}}(0) = C_1^T(t_f)$, and $\boldsymbol{x}_2^{\mathrm{adj}}$ and $\boldsymbol{y}_2^{\mathrm{adj}}$ are the state and the output adjoint response, respectively, for the initialization $\boldsymbol{x}_2^{\mathrm{adj}}(0) = C_2^T(t_f)$.



a) Original linear time-varying system with input u, output y, and initial condition x_0



b) Stochastic response using adjoint method

Fig. 2 Stochastic performance computation scheme based on adjoint response.

Proof. The proof is a slight variation of the proof of Proposition 3. The solution of Eq. (27) with initial value $P(0) = C_1^T(t_f)C_2(t_f)$ is

$$P(t_g) = \Phi^T(t_f, t_f - t_g)C_1^T(t_f)C_2(t_f)\Phi(t_f, t_f - t_g) \tag{36}$$

where Φ is the transition matrix function of system (1). Substituting solution (36) in Eq. (26), we deduce formula (35) using the same algebraic manipulations that were employed to deduce Eq. (33). □

The application of Proposition 4 to compute the covariance of two output variables is shown in Fig. 3. The blocks Bilinear Form stand for the computation of the bilinear form associated with the corresponding matrix on the two vector inputs. For example, the output of the block Bilinear Form Q_0 is $[\boldsymbol{x}_1^{\text{adj}}]^T Q_0 \boldsymbol{x}_2^{\text{adj}}$. In contrast to the classical scheme in Fig. 2, this computation requires the solution of two initial-value solutions, one for each output variable.

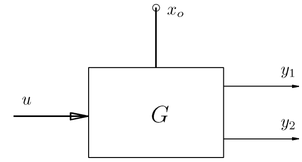
Naturally, in the case where the input is not white noise, the shaping filter approach can be used, and Propositions 3 and 4 apply to the augmented system obtained from the series connection of the original system with the shaping filter.

IV. Applications to Performance Evaluation of Homing Loops

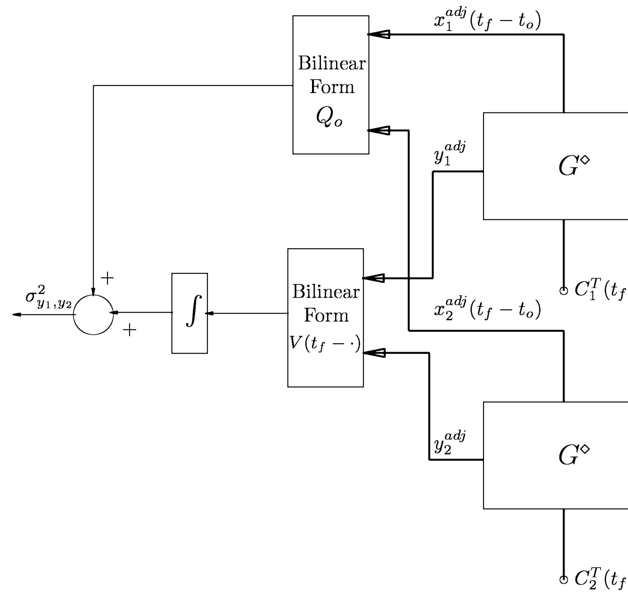
A. Miss Distance Average and Variance caused by Random Target Maneuver on Roll-Stabilized Missile

We take a new look at an example treated extensively in Ref. 2, Chap. 4. It concerns the standard homing loop using proportional navigation, and we assume that the target performs a step maneuver at a random moment of time. The start time of the maneuver is assumed to be uniformly distributed during the time of flight of the missile. We consider here two situations. First, we assume that the direction of the maneuver is also random, that is, the target can take a turn to the left or to the right with equal chances. This case is treated in Ref. 2. Subsequently, we assume that the direction of the maneuver is fixed and only the start time is random. To our knowledge, the treatment of this case has not appeared previously in the literature.

For completeness, we summarize here the derivation of the linear model of a two-dimensional engagement following Ref. 2, Chap. 2, for a homing missile using the proportional navigation guidance law. The linearization is performed around an ideal collision course defined by the target velocity, missile velocity and target course. The situation is shown in Fig. 4 and the following derivation uses the notation defined in Fig. 4. When it is assumed that the path angles



a) Original linear time-varying system



b) Covariance of two output signals using adjoint response

Fig. 3 Stochastic performance computation scheme based on adjoint response.

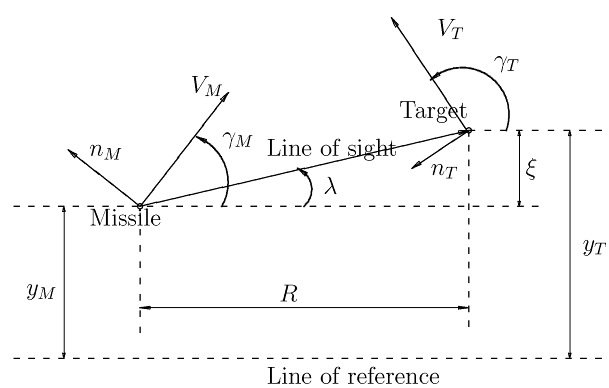


Fig. 4 Two-dimensional relative geometry of homing intercept.

γ_M and γ_T are small and close to 180 deg, respectively, then ξ, the lateral separation between missile and target, satisfies

$$\ddot{\xi} = n_T - n_M \tag{37}$$

where n_T is the lateral acceleration of the target and n_M is the lateral acceleration of the missile. The initial conditions for the second-order differential equation (37) are the initial lateral separation that is taken, by convention, to be null,

$$\xi(0) = 0 \tag{38}$$

and the first derivative is related to the initial heading error ε of the missile with respect to the collision course,

$$\dot{\xi}(0) = -V_M\varepsilon \tag{39}$$

where V_M is the missile velocity and the heading error is expressed in radians.

When it is assumed that the performance of the flight control system can be approximated with a first-order element with unity gain, the lateral acceleration of the missile satisfies

$$\dot{n}_M = (1/T)(-n_M + n_c) \tag{40}$$

where T is the missile time constant and n_c is the commanded lateral acceleration. In the case where the missile uses the proportional navigation guidance law,

$$n_c = N' V_c \dot{\lambda} \tag{41}$$

where N' is the effective navigation ratio, V_c is the closing velocity, and λ is the line-of-sight angle from the missile to the target with respect to an inertial reference. The closing velocity is assumed to be constant. Under the standing assumptions that the engagement is almost head-on, the closing velocity is $V_c = V_T + V_M$ and the line-of-sight angle can be written as

$$\lambda = \xi / R_{TM} \tag{42}$$

where R_{TM} is the relative distance between the missile and the target and can be expressed as

$$R_{TM} = V_c(t_F - t) \tag{43}$$

where t_F is the time of flight. Taking the time derivative in Eq. (42) and using Eq. (43) results in

$$\dot{\lambda} = \frac{\xi + \dot{\xi}(t_F - t)}{V_c(t_F - t)^2} \tag{44}$$

Combining Eqs. (37), (40), (41), and (44), we can write the model of the guidance loop as the following linear, time-varying state-space equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \dot{\xi}\\ n_M\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ \dfrac{N'}{T(t_F - t)^2} & \dfrac{N'}{T(t_F - t)} & -\dfrac{1}{T}\end{bmatrix}\begin{bmatrix}\xi\\ \dot{\xi}\\ n_M\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix} n_T \tag{45}$$

The state vector is in this case $\boldsymbol{x} = [\xi \;\; \dot{\xi} \;\; n_M]^T$ and the input is the target maneuver $\boldsymbol{u} = n_M$. The initial value of the state is

$$\boldsymbol{x}_0 = [0 \;\; -V_M\varepsilon \;\; 0]^T$$

The output of interest is in most cases the lateral separation at the end of the flight, $\xi(t_F)$, or the miss distance. To apply the adjoint method to the guidance loop in this case, the output is chosen to be

$$\boldsymbol{y} = [1 \;\; 0 \;\; 0]\begin{bmatrix}\xi\\ \dot{\xi}\\ n_M\end{bmatrix} \tag{46}$$

In the case where the missile lateral acceleration is the output of interest, the output is

$$\boldsymbol{y} = [0 \;\; 0 \;\; 1]\begin{bmatrix}\xi\\ \dot{\xi}\\ n_M\end{bmatrix} \tag{47}$$

It is easy to see that the adjoint response for the case where the output is given by Eq. (46) is the solution of the following initial-value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}^{\mathrm{adj}} = \begin{bmatrix}0 & 0 & \dfrac{N'}{Tt_g^2}\\ 1 & 0 & \dfrac{N'}{Tt_g}\\ 0 & -1 & -\dfrac{1}{T}\end{bmatrix}\boldsymbol{x}^{\mathrm{adj}}, \qquad \boldsymbol{x}^{\mathrm{adj}}(0) = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$y^{\mathrm{adj}} = [0 \;\; 1 \;\; 0]\boldsymbol{x}^{\mathrm{adj}} \tag{48}$$

According to Proposition 1, the miss distance due to a constant target maneuver is

$$\xi(t_F) = n_T \int_0^{(t_F - t_0)} y^{\mathrm{adj}}(\tau)\,\mathrm{d}\tau \tag{49}$$

In general, if the target maneuver follows an arbitrary time evolution, the miss distance due to the target maneuver is

$$\xi(t_F) = \int_0^{(t_F - t_0)} y^{\mathrm{adj}}(\tau) n_T(t_F - \tau)\,\mathrm{d}\tau \tag{50}$$

According to the same proposition, the miss distance due to the initial heading error is

$$\xi(t_F) = \left[\boldsymbol{x}^{\mathrm{adj}}(t_F - t_0)\right]^T \boldsymbol{x}_0 = V_M \varepsilon x_2^{\mathrm{adj}}(t_F - t_0) \tag{51}$$

With use of formula (50), it is easy to write the maximum miss distance that can be induced by a target whose lateral acceleration is limited by $n_{T,\max}$ as

$$\xi(t_F)_{\max} = n_{T,\max} \int_0^{(t_F - t_0)} |y^{\mathrm{adj}}(\tau)|\,\mathrm{d}\tau \tag{52}$$

The worst target maneuver is

$$n_T(\tau) = n_{T,\max}\,\mathrm{sign}\left[y^{\mathrm{adj}}(t_F - \tau)\right] \tag{53}$$

All of these formulas illustrate that with use of the adjoint response, it is easy to derive and numerically compute many different performance characteristics of the guidance loop.

Let us turn to the stochastic case. Although a homing loop may be characterized by many random signals, we shall look here only at the effect of the target lateral acceleration n_T. It is of little practical interest to consider the case where the target lateral acceleration is white (uncorrelated) noise. In fact, in most realistic situations, the target lateral acceleration at successive moments in time is relatively strongly correlated, even if it is not necessarily perfectly deterministic. Therefore, to apply the theory in Sec. III, it will be necessary to determine in each case a shaping filter.

As a first example of using the adjoint method in its state-space formulation for performance assessment of the guidance loop, we consider, following Ref. 2, Chap. 4, the case of a step target maneuver with random start time and random direction, that is,

$$n_T(t) = \pm n_{T,\max} u(t - t_s)$$

where t_s is a random variable, uniformly distributed within the interval $[0, t_F]$. (We assume here for simplicity that $t_0 = 0$.) It is shown in Ref. 2, Chap. 4, that a shaping filter for this stochastic process is the integrator

$$\frac{\mathrm{d}}{\mathrm{d}t} n_T = u_s \tag{54}$$

where u_s is a white noise signal of power spectral density $n_{T,\max}^2 / t_F$. The adjoint response should now be computed after adding the shaping filter equation (54) to the state-space model (45). The target lateral acceleration n_T now becomes part of the state vector, and the adjoint response of the augmented model is the solution to the following initial-value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}_s^{\mathrm{adj}} = \begin{bmatrix}0 & 0 & \dfrac{N'}{Tt_g^2} & 0\\ 1 & 0 & \dfrac{N'}{Tt_g} & 0\\ 0 & -1 & -\dfrac{1}{T} & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\boldsymbol{x}_s^{\mathrm{adj}}, \qquad \boldsymbol{x}_s^{\mathrm{adj}}(0) = \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$$

$$y_s^{\mathrm{adj}} = [0 \;\; 0 \;\; 0 \;\; 1]\boldsymbol{x}_s^{\mathrm{adj}} \tag{55}$$

It is easy to see that y_s^{adj} can be obtained directly by integrating the adjoint response y^{adj} obtained from problem (48). According to Proposition 3, the variance of the miss distance is

$$\sigma_\xi^2(t_F) = \frac{n_{T,\max}^2}{t_F}\int_0^{t_F}\left[y_s^{\text{adj}}(\tau)\right]^2 \mathrm{d}\tau$$

Because we assumed that the direction of the maneuver is also random, the average of the random input n_T is zero and, consequently, the average of the miss distance is also zero.

Let us consider now the case where the target maneuver has a fixed sign, but the starting time remains random and uniformly distributed. In this case, the average of the target lateral acceleration is not zero. In fact, it can be readily evaluated as

$$E[n_T(t)] = E[n_{T,\max}u(t-t_s)] = \int_0^{t_F} n_{T,\max}u(t-\tau)\,p_{t_s}(\tau)\,\mathrm{d}\tau$$

$$= \frac{n_{T,\max}}{t_F}\int_0^{t_F} u(t-\tau)\,\mathrm{d}\tau = \frac{n_{T,\max}}{t_F}\int_0^{t}\mathrm{d}\tau = \frac{n_{T,\max}t}{t_F} \qquad (56)$$

The response of the guidance loop to the average of n_T is the average of the miss distance, and according to formula (50) it can be computed using

$$E[\xi(t_F)] = \frac{n_{T,\max}}{t_F}\int_0^{t_F} y^{\text{adj}}(\tau)(t_F-\tau)\,\mathrm{d}\tau \qquad (57)$$

To compute the variance of the miss distance, we evaluate the centered correlation function n_T. We have, successively,

$$C_{n_T}^c(t_1,t_2) = E\left\{\left[n_T(t_1) - \frac{n_{T,\max}t_1}{t_F}\right]\left[n_T(t_2) - \frac{n_{T,\max}t_2}{t_F}\right]\right\}$$

$$= \int_0^{t_F}\left[n_{T,\max}u(t_1-\tau) - \frac{n_{T,\max}t_1}{t_F}\right]$$

$$\times\left[n_{T,\max}u(t_2-\tau) - \frac{n_{T,\max}t_2}{t_F}\right]p_{t_s}(\tau)\,\mathrm{d}\tau$$

$$= \frac{n_{T,\max}^2}{t_F}\left[\int_0^{t_F} u(t_1-\tau)u(t_2-\tau)\,\mathrm{d}\tau - \frac{t_1t_2}{t_F}\right]$$

$$= \frac{n_{T,\max}^2}{t_F}\left[\min(t_1,t_2) - \frac{t_1t_2}{t_F}\right]$$

$$= \frac{n_{T,\max}^2}{t_F^2}\min(t_1,t_2)[t_F - \max(t_1,t_2)] \qquad (58)$$

The last expression of the correlation function has the form postulated in the lemma in Appendix B, relation (B1) with $c_1(t) = (n_{T,\max}^2/t_F^2)t$ and $c_2(t) = t_F - t$. It is easy to see that both functions are positive in $[0, t_F]$ and that $c_1(t)/c_2(t) = (n_{T,\max}^2/t_F^2)[t/(t_F-t)]$ is strictly increasing in the same interval. Consequently, a shaping filter for this case is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_s = \frac{1}{t_F-t}u_s, \qquad n_T = (t_F-t)x_s \qquad (59)$$

where u_s is a white noise signal of power spectral density $n_{T,\max}^2/t_F^2$. The power spectral density of the white noise is a result of applying formula (B3). When the model of the guidance loop is augmented with the equations of the shaping filter (59), the adjoint response is, in this case, the solution of the initial-value problem,

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}_u^{\text{adj}} = \begin{bmatrix} 0 & 0 & \frac{N'}{Tt_g^2} & 0 \\ 1 & 0 & \frac{N'}{Tt_g} & 0 \\ 0 & -1 & -\frac{1}{T} & 0 \\ 0 & t_g & 0 & 0 \end{bmatrix}\boldsymbol{x}_u^{\text{adj}}, \qquad \boldsymbol{x}_u^{\text{adj}}(0) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$y_u^{\text{adj}} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{t_g}\end{bmatrix}\boldsymbol{x}_u^{\text{adj}} \qquad (60)$$

It is easy to see that y_u^{adj} can be obtained directly from the adjoint response y^{adj} in problem (48) by

$$y_u^{\text{adj}}(t_g) = \frac{1}{t_g}\int_0^{t_g}\tau y^{\text{adj}}(\tau)\,\mathrm{d}\tau$$

According to Proposition 3, the variance of the miss distance in this case is

$$\sigma_\xi^2(t_F) = \frac{n_{T,\max}^2}{t_F}\int_0^{t_F}\left[y_u^{\text{adj}}(\tau)\right]^2\mathrm{d}\tau \qquad (61)$$

For numerical testing, let us consider the case of a missile with time constant $T = 0.3$ s. The target is assumed to perform a maneuver of maximum $n_{T,\max} = 3\ g$, with random starting time. We consider both the case that the target maneuver has a random sign and the case that the sign is fixed. The effective navigation ratio is $N' = 3$. The average and standard deviation of the miss distance first were evaluated using the adjoint method, in the manner described in the preceding subsection. Subsequently, the result was checked using Monte Carlo simulations, that is, the linearized intercept model was run for several values of the time of flight t_F. For each value of the time of flight, 200 Monte Carlo runs were performed, each with a randomly generated start of the maneuver. For the achieved miss distances, the average and the standard deviation were computed using the familiar formulas. The final result is illustrated in Fig. 5 for the case where the sign of the target maneuver is random and in Fig. 6 for the case where the sign of the target maneuver is fixed.

It is clear that the numerical results obtained through the two methods are in good agreement. However, the Monte Carlo simulations took more than a quarter of an hour, whereas the adjoint response computation was finished in seconds.

Although the example discussed in this subsection demonstrates clearly the power of the adjoint method and illustrates the way in which the method is applied on state-space models, it does not make a clear case for the advantages of the state-space formulation. In the next subsection, the same homing scenario is pursued for the case of a rolling missile, and we shall see that, in this case, the state-space formulation leads to qualitatively better results than the classical formulation.

B. Miss Distance Average and Variance Caused by Random Target Maneuver on Rolling Missile

This example is taken from Shinar and Zarchan,[1] who apply the adjoint method to compute the miss distance of a rolling missile due to a deterministic target maneuver. We assume, as in Ref. 7, that the missile has two identical, noninteracting guidance loops that are operating in maneuver planes that are normal to each other. The missile is rotating about its longitudinal axis, that is, the intersection of the maneuver planes, with a roll rate that is known as a function of time. Inertial coupling and Magnus effects due to the roll rate are neglected. The relative geometry is assumed to satisfy in each of the maneuver planes the same assumptions as for the case of two-dimensional homing considered in the preceding subsection. In this case, the homing of the rolling missile can be modeled as two simultaneous and coupled two-dimensional homing loops in two

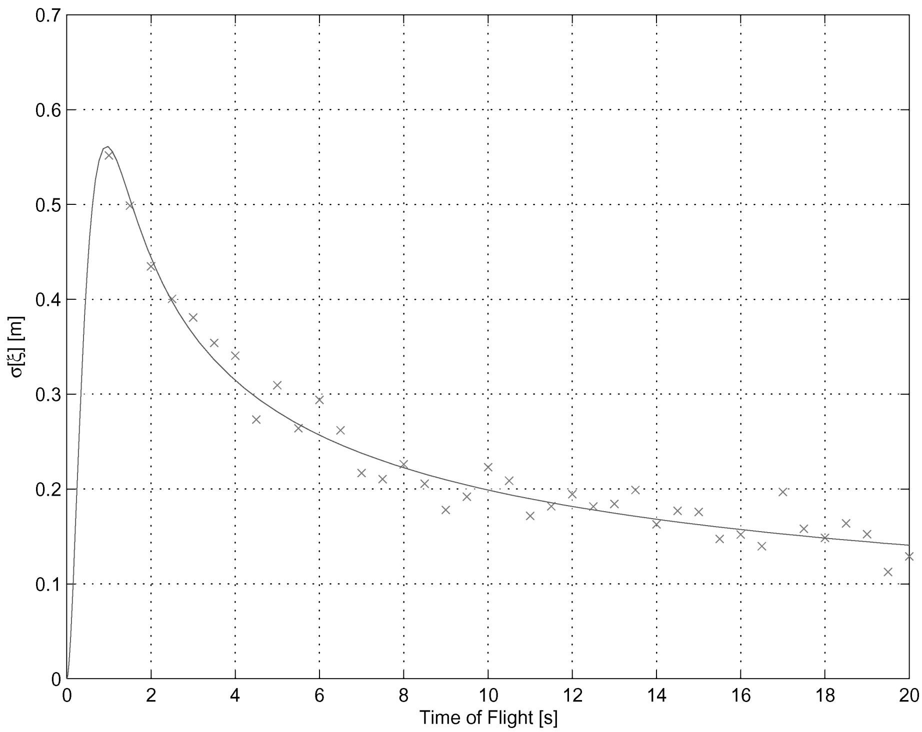


Fig. 5 Standard deviation of miss distance as function of time of flight, for target maneuver with random start time and random sign: ——, adjoint method and ×, Monte Carlo simulations.

perpendicular planes. Consequently, the performance of the rolling missile is best described in terms of the miss vector defined by the two miss distances realized in the two planes. We denote the two planes by y and z, and the correspondent miss vector components by ξ_y and ξ_z. With the earlier assumptions, they each satisfy an equation similar to Eq. (37):

$$\ddot{\xi}_y = n_{T,y} - n_{M,y} \tag{62}$$

$$\ddot{\xi}_z = n_{T,z} - n_{M,z} \tag{63}$$

where $n_{T,y}$ and $n_{T,z}$ are the components of the lateral acceleration of the target and $n_{M,y}$ and $n_{M,z}$ are the components of the lateral acceleration of the missile. For simplicity and without losing generality, we shall assume in the sequel that the target is only maneuvering in the y plane, so that $n_{T,z} = 0$. Because our analysis is limited to the response of the linearized intercept model, the effect of a target maneuver in an arbitrary direction can be directly obtained from the superposition principle. The lateral acceleration of the missile is produced in the body frame that is rolling. If we denote the components of the lateral acceleration of the missile with respect to the rolling body by $n_{M,1}$ and $n_{M,2}$, then we have

$$\begin{bmatrix} n_{M,y} \\ n_{M,z} \end{bmatrix} = \Phi(t) \begin{bmatrix} n_{M,1} \\ n_{M,2} \end{bmatrix} \tag{64}$$

where $\Phi(t)$ is the transformation matrix from the rolling coordinate frame to the nonrolling coordinate frame given by

$$\Phi(t) = \begin{bmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{bmatrix} \tag{65}$$

where $\phi(t)$ is the roll angle as a function of time that is assumed known. If the flight control system is modeled in the same way as in the case of the nonrolling missile, that is, a first-order transfer function with unity gain, then we have

$$\dot{n}_{M,i} = (1/T)(-n_{M,i} + n_{c,i}), \qquad i = 1, 2 \tag{66}$$

where $n_{c,1}$ and $n_{c,2}$ are the commanded accelerations in the two guidance channels. Because both guidance channels use the same proportional navigation guidance law,

$$n_{c,i} = N' V_c \dot{\lambda}_i, \qquad i = 1, 2 \tag{67}$$

where $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are the components of the line-of-sight angular velocity that are provided by the target tracking system of the missile in the rolling-body coordinate frame. They are related to the components of the same vector in the nonrolling coordinate frame through the inverse transformation of the one applied in matrix (64):

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \Phi^T(t) \begin{bmatrix} \dot{\lambda}_y \\ \dot{\lambda}_z \end{bmatrix} \tag{68}$$

Here we used the fact that the matrix Φ in Eq. (65) is orthogonal and, therefore, its transpose is its inverse. The components $\dot{\lambda}_y$ and $\dot{\lambda}_z$ of the angular rate of the line of sight in the nonrolling coordinate frame satisfy each Eq. (44) in their respective plane. That means

$$\dot{\lambda}_y = \frac{\xi_y + \dot{\xi}_y(t_F - t)}{V_c(t_F - t)^2}, \qquad \dot{\lambda}_z = \frac{\xi_z + \dot{\xi}_z(t_F - t)}{V_c(t_F - t)^2} \tag{69}$$

When Eqs. (62–64) and (66–69) are combined, the state differential equation of the linear intercept model can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = A(t)\boldsymbol{x} + B\boldsymbol{u} \tag{70}$$

where the state vector and the input are

$$\boldsymbol{x} = [\xi_y \quad \dot{\xi}_y \quad \xi_z \quad \dot{\xi}_z \quad n_{M,1} \quad n_{M,2}]^T, \qquad \boldsymbol{u} = n_{T,y} \tag{71}$$

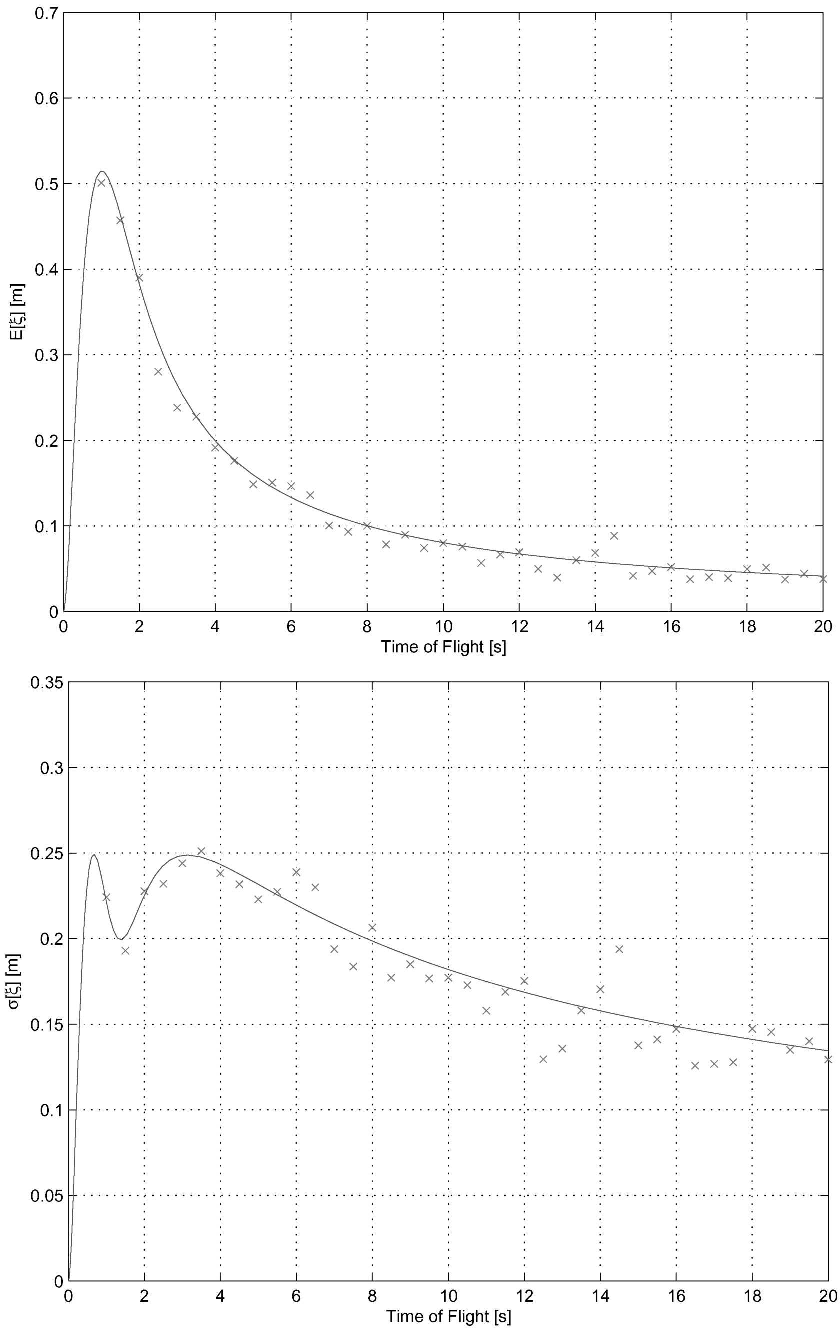


Fig. 6 Average and standard deviation of the miss distance as a function of time of flight, target maneuver with random start time and fixed sign: ——, adjoint method and × Monte Carlo simulations.

and the system matrices are

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\phi(t) & -\sin\phi(t) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi(t) & \cos\phi(t) \\ \dfrac{N'\cos\phi(t)}{T(t_F-t)^2} & \dfrac{N'\cos\phi(t)}{T(t_F-t)} & \dfrac{N'\sin\phi(t)}{T(t_F-t)^2} & \dfrac{N'\sin\phi(t)}{T(t_F-t)} & -\dfrac{1}{T} & 0 \\ -\dfrac{N'\sin\phi(t)}{T(t_F-t)^2} & -\dfrac{N'\sin\phi(t)}{T(t_F-t)} & \dfrac{N'\cos\phi(t)}{T(t_F-t)^2} & \dfrac{N'\cos\phi(t)}{T(t_F-t)} & 0 & -\dfrac{1}{T} \end{bmatrix} \tag{72}$$

$$B = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T \tag{73}$$

The outputs of interest are the two components of the miss vector:

$$y_1 = \xi_y = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\,\boldsymbol{x} \tag{74}$$

$$y_2 = \xi_z = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]\,\boldsymbol{x} \tag{75}$$

If we need to analyze the response of the homing loop to a target maneuver in the y plane represented as a step in the lateral acceleration of magnitude $n_{T,\max}$ with random starting time and random sign as in the preceding subsection, the state-space model needs to be extended with Eq. (54). In this case, the state and input vectors become

$$\boldsymbol{x}_s = [\xi_y \quad \dot{\xi}_y \quad \xi_z \quad \dot{\xi}_z \quad n_{M,1} \quad n_{M,2} \quad n_{T,y}]^T, \qquad \boldsymbol{u}_s = u_s \tag{76}$$

where u_s is a white noise signal of power spectral density $n_{T,\max}/t_F$ and the state equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_s = A_s(t)\boldsymbol{x}^{\mathrm{ext}} + B_s\boldsymbol{u}^{\mathrm{ext}} \tag{77}$$

with

$$A_s(t) = \begin{bmatrix} & & & & & & 0 \\ & & & & & & 1 \\ & & & & & & 0 \\ & & & A(t) & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{78}$$

The adjoint response $\boldsymbol{y}_y^{\mathrm{adj}}$ for this system corresponding to the miss distance in the y plane is the solution to the initial-value problem,

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}_s^{\mathrm{adj}} = A_s^T(t_F - t_g)\boldsymbol{x}_s^{\mathrm{adj}}$$

$$\boldsymbol{x}_s^{\mathrm{adj}}(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \qquad \boldsymbol{y}_{s,y}^{\mathrm{adj}} = B_s^T\boldsymbol{x}^{\mathrm{adj}}$$

whereas the adjoint response $\boldsymbol{y}_{s,z}^{\mathrm{adj}}$ corresponding to the miss distance in the z plane satisfies the same relations, except that the initial value changes to

$$\boldsymbol{x}_s^{\mathrm{adj}}(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

The covariance matrix of the miss vector follows from Propositions 3 and 4:

$$P = \frac{n_{T,\max}^2}{t_F}\int_0^{t_F}\begin{bmatrix} \left[\boldsymbol{y}_{s,y}^{\mathrm{adj}}(\tau)\right]^2 & \boldsymbol{y}_{s,y}^{\mathrm{adj}}(\tau)\boldsymbol{y}_{s,z}^{\mathrm{adj}}(\tau) \\ \boldsymbol{y}_{s,y}^{\mathrm{adj}}(\tau)\boldsymbol{y}_{s,z}^{\mathrm{adj}}(\tau) & \left[\boldsymbol{y}_{s,z}^{\mathrm{adj}}(\tau)\right]^2 \end{bmatrix}\mathrm{d}\tau \tag{79}$$

Notice that the classical treatment would only provide the elements on the diagonal of the matrix P, but not the off-diagonal elements. A simple numerical example will demonstrate that the off-diagonal elements bring a useful contribution to the prediction of the distribution of the miss vector.

The case that the sign of the lateral acceleration of the target is fixed is treated identically to the case of the roll-stabilized missile. The average of the miss vector is

$$E[\xi_i(t_F)] = \frac{n_{T,\max}}{t_F}\int_0^{t_F} y_i^{\mathrm{adj}}(\tau)(t_F - \tau)\,\mathrm{d}\tau, \qquad i = y, z \tag{80}$$

where y_y^{adj} and y_z^{adj} are the adjoint responses of the system (70) with outputs (74) and (75).

To obtain the covariance matrix, the state equations (70) are extended with the equations of the shaping filter (59) that results in the new system matrices,

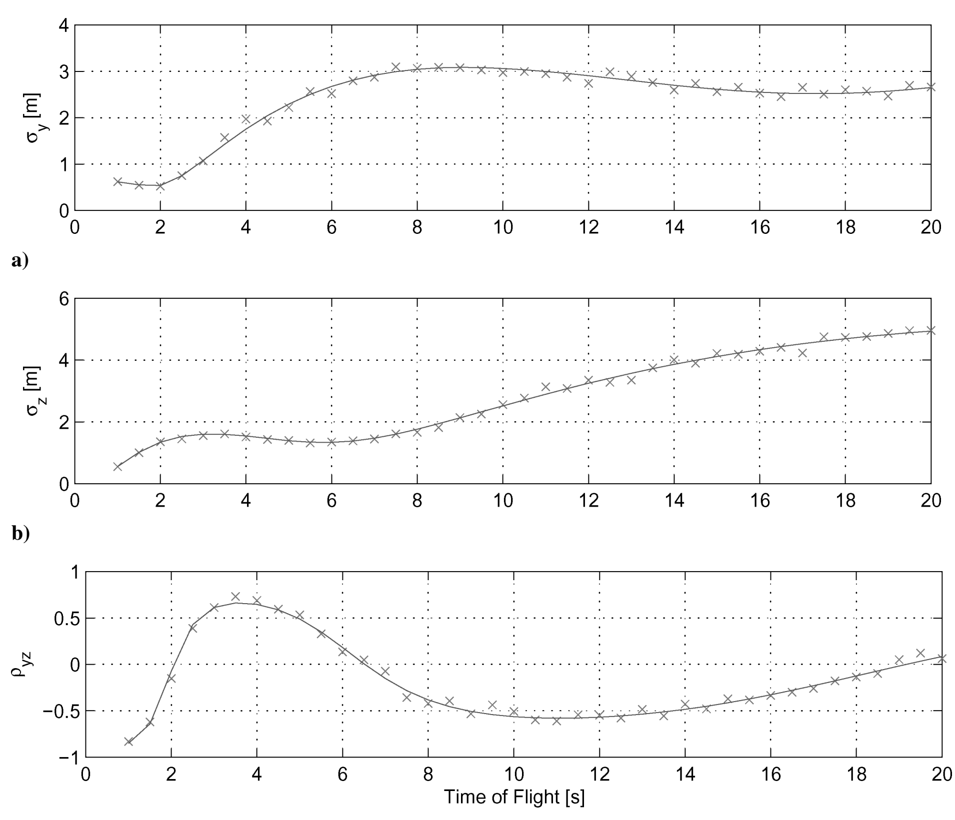


Fig. 7 Standard deviations and correlation coefficient for components of miss vector as function of time of flight, for rolling missile against target with maneuver with a random start time and random sign: ——, adjoint method and ×, Monte Carlo simulations.

$$A_u(t) = \begin{bmatrix} & & & & & & 0 \\ & & & & & & t_F - t \\ & & & & & & 0 \\ & & A(t) & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{t_F - t} \end{bmatrix} \tag{81}$$

The adjoint response $y_{u,y}^{\text{adj}}$ is obtained by solving the initial-value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\boldsymbol{x}_u^{\text{adj}} = A_u^T(t_F - t_g)\boldsymbol{x}_u^{\text{adj}}$$

$$\boldsymbol{x}_u^{\text{adj}}(0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \qquad \boldsymbol{y}_{u,y}^{\text{adj}} = B_u^T \boldsymbol{x}^{\text{adj}}$$

whereas for the adjoint response $y_{u,z}^{\text{adj}}$, the initial condition becomes

$$\boldsymbol{x}_u^{\text{adj}}(0) = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

The variance matrix of the miss distance is computed with formula (79), where $y_{u,y}^{\text{adj}}$ and $y_{u,z}^{\text{adj}}$ are used instead of $y_{s,z}^{\text{adj}}$ and $y_{s,z}^{\text{adj}}$.

For a numerical test of these results we consider a missile with the same dynamic response given by the time constant $T = 0.3$ s as in the preceding subsection, but that is rolling around its longitudinal axis with a constant angular rate $\omega = \pi$ rad/s, such that $\phi(t) = \omega t$. The target performs a maneuver in the y plane with $n_{T,\max} = 3$ g. Just as in the case of the roll-stabilized missile, we compare the results obtained using the adjoint method with the results obtained using Monte Carlo simulations with 200 runs for each value of the time of flight, on the linearized model.

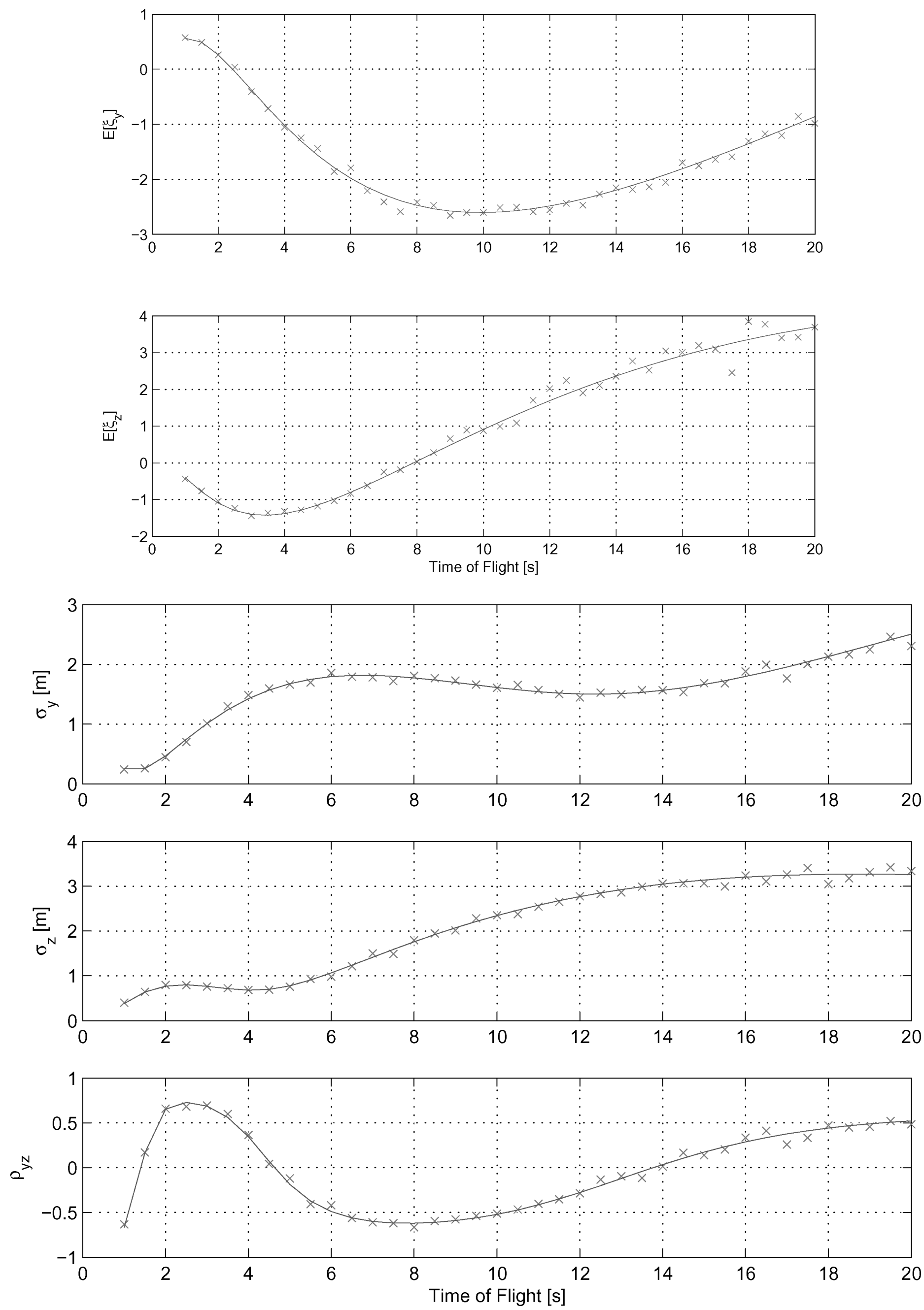


Fig. 8 Average, standard deviations and correlation coefficient of the components of miss vector as function of time of flight, for rolling missile against target maneuver with random start time and fixed sign: ——, adjoint method and ×, Monte Carlo simulations.

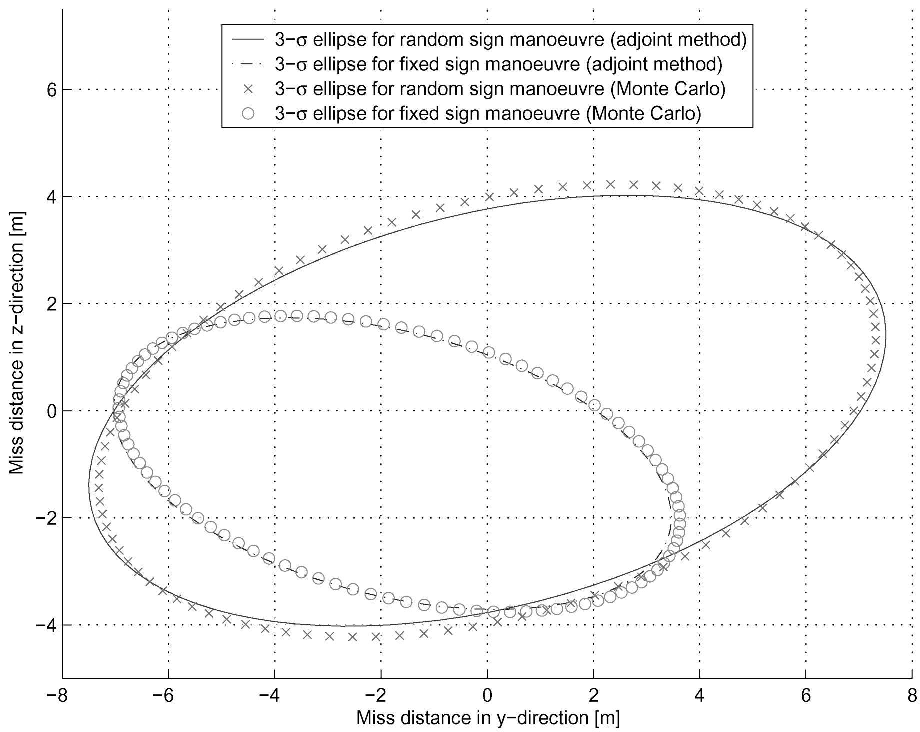


Fig. 9 Estimated miss vector distribution using Monte Carlo simulation and the adjoint method represented as 3-sigma ellipses for the case of a rolling missile against a target maneuver with random start time: —— and ×, random sign and – – – and ○, fixed sign.

For the case that the target maneuver has random start time and random sign, the results of the Monte Carlo and the adjoint method are shown in Fig. 7. Figures 7a, 7b, and 7c present the standard deviations of the miss distances in the y and z directions and the correlation coefficient between these miss distances. Remember that the correlation coefficient is

$$\rho_{yz} = \sigma_{yz}/\sigma_y\sigma_z$$

where σ_{yz} is the covariance of the two miss distances. The correlation coefficient takes values between -1 and 1, and, if it is zero, then the two variables are uncorrelated. Figure 7c indicates that for most times of flight, the two variables are quite strongly correlated. Notice also that the results obtained using Monte Carlo simulations agree quite well with the results obtained using the adjoint method.

For the case where the target maneuver has a fixed sign, the results of the Monte Carlo and the adjoint method are shown in Fig. 8.

It is interesting to compare the estimated miss vector distributions based on 3-sigma ellipses for the two cases. These ellipses are defined by the inequality

$$(x - \bar{x})P^{-1}(x - \bar{x}) \leq 3$$

where P is the variance matrix and $\bar{x}$ is the average vector. These ellipses are represented for a single value of the time of flight (5.5 s) in Fig. 9 both for the case that the maneuver sign is random and for the case that the sign is fixed.

V. Conclusions

The adjoint method is known as a powerful tool for preliminary evaluation of guidance-loop performance. Most references present this method in an input–output setting. We gave here a general presentation of the method in a state-space context. It turns out that, not only is this an elegant way of presenting the method, but it lead us to significant extensions of the capabilities that the adjoint method offers for performance evaluation of guidance systems under random exogenous signals. In particular, the method can be extended to the case where there are several input signals that are correlated, or where there are several output signals of interest and it is necessary to compute not only the covariance of each output separately, but the entire variance matrix of the output vector. This second situation was illustrated with a simple example involving a rolling missile against a maneuvering target. The generalized adjoint method was used to compute the variance matrix of the miss vector. It turned out that the components of the miss vector are correlated and the generalized adjoint method is capable of computing the correlation coefficient between the two components with high accuracy while avoiding Monte Carlo simulations.

The results were all presented and proved for the continuous-time case. In Appendix A, the corresponding discrete-time results are stated. It is also possible to extend the theory to hybrid (sample-data) systems as well, but this further extension will be reported elsewhere.

Appendix A: Discrete-Time Case

This section presents the discrete-time counterparts of the results in Secs. II and III. For conciseness, proofs have been omitted. Let us consider a linear time-varying system G with m inputs and p outputs:

$$\boldsymbol{x}_{k+1} = A(k)\boldsymbol{x}_k + B(k)\boldsymbol{u}_k, \qquad k = k_0, k_0 + 1, \ldots$$
$$\boldsymbol{y}_k = C(k)\boldsymbol{x}_k \tag{A1}$$

with initial condition

$$\boldsymbol{x}_{k_0} = \boldsymbol{x}_0 \tag{A2}$$

Here $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are assumed to be matrix-valued functions defined on (a subset of the) integers with $A(k) \in R^{n\times n}$, $B(k) \in R^{n\times m}$, and $C(k) \in R^{p\times n}$. Assume that we are interested in the value of the output of the system at the moment k_f.

The dual of the system (A1) at $k_f \in N$ is defined as the linear discrete time-varying system with p input and m outputs:

$$\boldsymbol{x}^{\diamond}_{k_g} = A^T(k_f - k_g)\boldsymbol{x}^{\diamond}_{k_g-1} + C^T(k_f - k_g)\boldsymbol{u}^{\diamond}_{k_g-1}, \qquad k_g = 1, 2, \ldots$$
$$\boldsymbol{y}^{\diamond}_{k_g-1} = B^T(k_f - k_g)\boldsymbol{x}^{\diamond}_{k_g-1} \tag{A3}$$

Assume that system (A1) has a single output $p = 1$. The adjoint response of system (A1) at k_f is defined as the free response ($\boldsymbol{u}^{\diamond} \equiv 0$) of the dual system with initial condition

$$\boldsymbol{x}^{\diamond}_0 = C^T(k_f) \tag{A4}$$

We denote by $\boldsymbol{x}^{\text{adj}}$ the state adjoint response and by $\boldsymbol{y}^{\text{adj}}$ the output adjoint response. By definition, they satisfy

$$\boldsymbol{x}_{k_g}^{\text{adj}} = A^T(k_f - k_g)\boldsymbol{x}_{k_g-1}^{\text{adj}}, \qquad k_g = 1, 2, \ldots$$

$$\boldsymbol{y}_{k_g-1}^{\text{adj}} = B^T(k_f - k_g)\boldsymbol{x}_{k_g-1}^{\text{adj}}, \qquad x_0^{\text{adj}} = C^T(k_f) \tag{A5}$$

The following results are the discrete-time counterparts of Propositions 1–4.

Proposition A1. The final value of the output of system (A1), with initial condition (A2) is

$$\boldsymbol{y}_{k_f} = \left[\boldsymbol{x}_{(k_f-k_0)}^{\text{adj}}\right]^T \boldsymbol{x}_0 + \sum_{k=k_0}^{k_f-1} \left[\boldsymbol{y}_{(k_f-k-1)}^{\text{adj}}\right]^T \boldsymbol{u}_k \tag{A6}$$

where $\boldsymbol{x}^{\text{adj}}$ and $\boldsymbol{y}^{\text{adj}}$ are the state and the output of the adjoint system at k_f, respectively.

Proposition A2. Let the linear discrete time-varying system with two output signals

$$\boldsymbol{x}_{k+1} = A(k)\boldsymbol{x}_k + B(k)\boldsymbol{u}_k, \qquad y_{1,k} = C_1(k)\boldsymbol{x}_k$$

$$y_{2,k} = C_2(k)\boldsymbol{x}_k \tag{A7}$$

$C_1(k)$ and $C_2(k)$ are in $R^{1\times n}$, each u_k is a stochastic variable of zero mean and covariance matrix V_k, and each pair u_k, u_s with $k \neq s$ is uncorrelated. Assume also that the initial value

$$\boldsymbol{x}_{k_0} = \boldsymbol{x}_0 \tag{A8}$$

is a stochastic vector of zero mean and covariance matrix Q_0.

The covariance of y_{1,k_f} and y_{2,k_f} is

$$\sigma_{y_1y_2}(k_f) = \operatorname{tr} P_{k_f} Q_0 + \sum_{k=k_0}^{k_f-1} \operatorname{tr}\left[B^T(k-1)P_{k_f-k-1}B(k-1)V_k\right] \tag{A9}$$

where P_k is an $n \times n$ matrix sequence satisfying the recursion

$$P_{k_g} = A^T(k_f - k_g)P_{k_g-1}A(k_f - k_g) + P_{k_g-1} \tag{A10}$$

with initial condition

$$P_0 = C_1^T(k_f)C_2(k_f) \tag{A11}$$

In particular, if $y_1 = y_2 = y$, that is, $C_1(k) = C_2(k)$ for all k, formula (A9) gives the expression of the variance of the output y_{k_f}.

In the case where $V_k = V$ const for each k, formula (A9) becomes

$$\sigma_{y_1y_2}(k_f) = \operatorname{tr} P_{k_f} Q_0 + \operatorname{tr}\left[V \sum_{k=k_0}^{k_f-1} B^T(k-1)P_{k_f-k-1}B(k-1)\right] \tag{A12}$$

Proposition A3. Consider system (A1) with a single output variable [$C(k)$ is in $R^{1\times n}$]. The variance of the output at time k_f is

$$\sigma_y^2(k_f) = \left[\boldsymbol{x}_{k_f-k_0}^{\text{adj}}\right]^T Q_0 \boldsymbol{x}_{k_f-k_0}^{\text{adj}} + \sum_{k=k_0}^{k_f-1}\left[\boldsymbol{y}_{k_f-k-1}^{\text{adj}}\right]^T V_k \boldsymbol{y}_{k_f-k-1}^{\text{adj}} \tag{A13}$$

where $\boldsymbol{x}^{\text{adj}}$ and $\boldsymbol{y}^{\text{adj}}$ are the state and the output adjoint response of system (A1) at k_f.

Proposition A4. The covariance of the two output signals of the system (A7) at time k_f is

$$\sigma_{y_1,y_2}(k_f) = \left[\boldsymbol{x}_{1,k_f-k_0}^{\text{adj}}\right]^T Q_0 \boldsymbol{x}_{2,k_f-k_0}^{\text{adj}} + \sum_{k=k_0}^{k_f-1}\left[\boldsymbol{y}_{1,k_f-k-1}^{\text{adj}}\right]^T V_k \boldsymbol{y}_{2,k_f-k-1}^{\text{adj}} \tag{A14}$$

where $\boldsymbol{x}_1^{\text{adj}}$ and $\boldsymbol{y}_1^{\text{adj}}$ are the state and the output adjoint response for the initialization $\boldsymbol{x}_1^{\text{adj}}(0) = C_{1,k_f}^T$, respectively, and $\boldsymbol{x}_2^{\text{adj}}$ and $\boldsymbol{y}_2^{\text{adj}}$ are the state and the output adjoint response for the initialization $\boldsymbol{x}_2^{\text{adj}}(0) = C_{2,k_f}^T$, respectively.

Appendix B: Time-Varying Shaping Filter

In Sec. IV, we use the following auxiliary result for constructing of the shaping filter.

Lemma. Let us assume that the autocovariance function of a given stochastic process y can be factorized as

$$R_y(t_1, t_2) = c_1[\min(t_1, t_2)]c_2[\max(t_1, t_2)] \tag{B1}$$

where c_1 and c_2 are two differentiable functions, positive in the interval of interest and such that c_1/c_2 is strictly increasing in the same interval. Consider the time-varying linear system

$$\dot{x} = g(t)u, \qquad x(0) = 0, \qquad y = [1/g(t)]x \tag{B2}$$

where $g(t) = 1/c_2(t)$. If u is a zero mean white noise process of power spectral density

$$\Phi_u(t) = c_1'(t)c_2(t) - c_1(t)c_2'(t) \tag{B3}$$

then the autocovariance function of the output of system (B2) coincides with R_y in factorization (B1)

Notice that because we assumed that c_1/c_2 is strictly increasing, it follows that Φ_u is strictly positive, and so it can indeed be the power spectral density of a white noise process.

Applying relation (16) to system (B2) [with $A(t) \equiv 0$, $B(t) = g(t)$, and $C(t) = 1/g(t)$], we see that the variance of $x(t)$ is satisfying

$$\dot{Q} = g^2(t)\Phi_u(t) = \frac{c_1'(t)c_2(t) - c_1(t)c_2'(t)}{c_2^2(t)}$$

We deduce that $Q(t) = c_1(t)/c_2(t)$. Furthermore, using formula (19), and taking into account that the transition matrix is in this case the identity, we have

$$R_x(t_1, t_2) = C_x(t_1, t_2) = \begin{cases} \dfrac{c_1(t_1)}{c_2(t_1)}, & t_2 \geq t_1 \\[2ex] \dfrac{c_1(t_2)}{c_2(t_2)}, & t_1 \geq t_2 \end{cases}$$

Now the autocovariance function of the output of system (B2) is

$$R_y(t_1, t_2) = \frac{1}{g(t_1)}\frac{1}{g(t_2)}C_x(t_1, t_2) = \begin{cases} c_1(t_1)c_2(t_2), & t_2 \geq t_1 \\ c_1(t_2)c_2(t_1), & t_1 \geq t_2 \end{cases} \tag{B4}$$

which obviously coincides with factorization (B1).

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